NUMERICAL SOLUTION OF FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS BY SPECTRAL METHODS

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A thesis submitted in
fulfillment of the requirement for the award of
Doctor of Philosophy in Science

Faculty of Applied Sciences and Technology
Universiti Tun Hussein Onn Malaysia

SEPTEMBER 2019

I would like to dedicate my Doctoral thesis to Prof. Madya Dr. Phang Chang and my beloved parents whose sincere prayers make it possible for me to fulfill their utmost desire. May Allah always bless them with more happiness and good health.

ACKNOWLEDGEMENT

I would like to thank Allah Almighty, whose benediction bestowed upon me talented teachers, provided me sufficient opportunities and enabled me to undertake and execute this research work.

First of all, I want to express my deepest gratitude to my worthy, affectionate, kind and most respected supervisor Prof. Madya Dr. Phang Chang who pointed me to the exciting subject of Fractional Partial Differential equations and for his excellent guidance and encouragement during the period of this research. His patience, inspiration and guidance always lead me to positive outcomes and guided me in a prospective direction whenever I found myself caught in a bottleneck situation. I owe a great debt to him for his constructive comments on my work throughout this journey.

Without his guidance, I would not be able to face the challenges, overcome various obstacles and hardship during these years and eventually accomplish the results obtained. My special thanks are due to my sponsor, Universiti Tun Hussein Onn Malaysia for granting me the scholarship. This greatly helps me to focus fully on my research without having to worry over financial difficulties. I will never forget the educational facilities and research oriented environment provided by the Faculty of Applied Sciences and Technology (FAST) and the Universiti Tun Hussein Onn Malaysia (UTHM).

Special thanks to my friend, Dr. Loh Jian Rong, who inspires me with many ideas through many fruitful discussions. Discussions and ideas by him continue to motivate and inspires me with new possibilities.

Finally, my heartiest and warm thanks to my husband, daughters, parents and family, who deserves great thanks for their immense support throughout my PhD time which guided me to face various problems and dilemma in both personal life and research.

ABSTRACT

Fractional partial differential equations (FPDEs) have become essential tool for the modeling of physical models by using spectral methods. In the last few decades, spectral methods have been developed for the solution of time and space dimensional FPDEs. There are different types of spectral methods such as collocation methods, Tau methods and Galerkin methods. This research work focuses on the collocation and Tau methods to propose an efficient operational matrix methods via Genocchi polynomials and Legendre polynomials for the solution of two and three dimensional FPDEs. Moreover, in this study, Genocchi wavelet-like basis method and Genocchi polynomials based Ritz- Galerkin method have been derived to deal with FPDEs and variable- order FPDEs. The reason behind using the Genocchi polynomials is that, it helps to generate functional expansions with less degree and small coefficients values to derive the operational matrix of derivative with less computational complexity as compared to Chebyshev and Legendre Polynomials. The results have been compared with the existing methods such as Chebyshev wavelets method, Legendre wavelets method, Adomian decomposition method, Variational iteration method, Finite difference method and Finite element method. The numerical results have revealed that the proposed methods have provided the better results as compared to existing methods due to minimum computational complexity of derived operational matrices via Genocchi polynomials. Additionally, the significance of the proposed methods has been verified by finding the error bound, which shows that the proposed methods have provided better approximation values for under consideration FPDEs.

ABSTRAK

Persamaan Pembezaan Separa Pecahan (PPSP) telah menjadi alat penting untuk pemodelan model fizikal dengan menggunakan kaedah spektral. Dalam beberapa dekad yang lalu, kaedah spektral telah dibangunkan untuk penyelesaian PPSP bagi terbitan dimensi masa dan ruang. Terdapat pelbagai jenis kaedah spektral seperti kaedah kolokasi, kaedah Tau dan kaedah Galerkin. Kajian ini memberi tumpuan kepada kaedah kolokasi dan kaedah Tau untuk mencadangkan kaedah matriks operasi yang berkesan melalui polinomial Genocchi dan polynomial Legendre untuk penyelesaian dua dan tiga dimensi PPSP. Tambahan pula, dalam kajian ini, kaedah asas seperti wavelet Genocchi dan kaedah Ritz-Galerkin berasaskan polynomial Gennochi telah diperolehi untuk menangani PPSP dan PPSP peringkat pembolehubah . Alasan di sebalik menggunakan polinomial Genocchi adalah bahawa ia membantu untuk menghasilkan kembangan fungsi dengan nilai pekali yang kecil dan cara memperoleh matriks operasi pembezaan yang kurang rumit pengiraannya berbanding dengan Polynomial Chebyshev dan Legendre. Hasilnya telah dibandingkan dengan kaedah yang sedia ada seperti kaedah wavelet Chebyshev, kaedah wavelet Legendre, kaedah penguraian Adomian, kaedah lelaran variasi, kaedah perbezaan terhingga dan kaedah unsur terhingga. Keputusan berangka telah mendedahkan bahawa kaedah yang dicadangkan telah memberikan hasil yang lebih baik berbanding dengan kaedah yang sedia ada disebabkan oleh pengiraan matriks operasi adalah kurang rumit dengan polinomial Genocchi. Selain itu, kepentingan kaedah yang dicadangkan telah dibukti dengan ralat sempadan, yang menunjukkan bahawa kaedah yang dicadangkan telah memberikan nilai anggaran yang lebih baik untuk PPSP.

TABLE OF CONTENTS

	TITLE	**i**
	DECLARATION	**ii**
	DEDICATION	**iii**
	ACKNOWLEDGEMENT	**iv**
	ABSTRACT	**v**
	ABSTRAK	**vi**
	TABLE OF CONTENTS	**vii**
	LIST OF TABLES	**xiii**
	LIST OF FIGURES	**xviii**
	LIST OF SYMBOLS AND ABBREVIATIONS	**xix**
	LIST OF APPENDICES	**xxiii**
	LIST OF PUBLICATIONS	**xxiv**
CHAPTER 1	**INTRODUCTION**	**1**
	1.1 Preliminaries	1
	1.1.1 Fractional partial differential equations	1
	1.1.2 Mathematical solution	3
	1.1.3 Solution methods	3
	1.1.4 Method of weighed residuals	3
	1.1.5 Spectral methods	4
	1.2 Some basic concepts	4
	1.2.1 Polynomials	5
	1.2.2 Fractional derivative	5
	1.2.3 Operational matrix	6
	1.3 Problem statements	6
	1.4 Research aim and objectives	8
	1.5 Scope of research	8

	1.6	Main contribution	9
	1.7	Thesis organization	9
CHAPTER 2	**LITERATURE REVIEW**		**10**
	2.1	Introduction	10
	2.2	Wavelets theory and some of its basic definitions	10
		2.2.1 Wavelets	11
	2.3	Numerical methods for solving FDEs and FPDEs	12
		2.3.1 Legendre polynomials and Legendre wavelets method	12
		2.3.2 Chebyshev polynomials and Chebyshev wavelets method	16
		2.3.3 Bernoulli polynomials and Bernoulli wavelets method	18
		2.3.4 Spectral Collocation method	19
		2.3.5 Tau method	21
	2.4	Analytical methods for solving FDEs and FPDEs	23
	2.5	Numerical and analytical methods for two and three dimensional FPDEs	24
	2.6	Chapter summary	25
CHAPTER 3	**RESEARCH METHODOLOGY**		**26**
	3.1	Preliminaries	26
		3.1.1 Euler's Gamma function	27
		3.1.2 Fractional derivatives and integrals	27
		3.1.3 Riemann-Liouville fractional integral and derivative	27
		3.1.4 Basic properties of Riemann-Liouville integration	28
		3.1.5 Caputo's fractional derivative	29
		3.1.6 Basic properties of Caputo derivative	29
		3.1.7 Some basic definitions of variable order fractional operators	30

		3.1.8 Satisfier function	32
		3.1.9 Basic definitions of Bernoulli numbers and Bernoulli polynomials	32
		3.1.10 Basic definitions of Euler numbers and Euler polynomials	33
	3.2	Introduction to spectral methods	34
	3.3	Analysis of methods for the solution of FPDEs	36
		3.3.1 Two-dimensional FPDEs via polynomials operational matrix	36
		3.3.2 Three-dimensional FPDEs via polynomials operational matrix	38
		3.3.3 FPDEs with variable coefficients via polynomials operational matrix	41
		3.3.4 FPDEs via wavelet-like function	44
		3.3.5 VOFPDEs via wavelet-like function	46
		3.3.6 FPDEs via Ritz-Galerkin method	47
	3.3	Finding error	48
	3.4	Chapter summary	49
CHAPTER 4	**COLLOCATION METHOD FOR SOLVING FPDES OF TWO AND THREE DIMENSI-ONS USING GENOCCHI POLYNOMIALS OPERATIONAL MATRIX**		**51**
	4.1	Introduction	51
	4.2	Definition and properties of Genocchi polynomials	52
		4.2.1 Shifted Chebyshev Gauss Lobatto (CGL) nodes	53
	4.3	Function approximation of Genocchi polynomials	57
		4.3.1 Genocchi polynomials operational matrix of Caputo's fractional differentiation	64
	4.4	Error analysis by using shifted CGL nodes	65

4.5 Application of Genocchi polynomials operational matrix for two-dimensional time-fractional diffusion-wave equation with damping 67

4.5.1 Numerical results 68

4.6 Application of Genocchi polynomials for two-dimensional time-fractional convection diffusion equation with variable coefficient 72

4.6.1 Numerical results 73

4.7 Application of Genocchi polynomials for three-dimensional time-fractional diffusion-wave equation 78

4.7.1 Numerical results 79

4.8 Application of Genocchi polynomials for three-dimesional time-fractional modified anamalous sub-diffusion equation 80

4.8.1 Numerical results 82

4.9 Chapter summary 84

CHAPTER 5 LEGENDRE POLYNOMIALS OPERATIONAL MATRIX FOR SOLVING FPDES WITH VARIABLE COEFFICIENT 85

5.1 Introduction 85

5.2 Preliminaries: Recurrence relation for shifted Legendre polynomials 85

5.3 Function approximation of shifted Legendre polynomials 86

5.4 Legendre polynomial operational matrix of fractional derivative and integration 89

5.4.1 Legendre polynomial operational matrix of fractional derivative 89

5.4.2 Legendre polynomial operational matrix of fractional integration 90

5.5 Application of Legendre polynomials for two-dimensional time-fractional convection

		diffusion equation with variable coefficient	94
		5.5.1 Numerical results	95
	5.6	Application of Legendre polynomials for two- dimensional time-fractional wave equation	97
		5.6.1 Numerical results	97
	5.7	Application of Legendre polynomials for two-dimensional time-fractional diffusion equation with variable coefficient	99
		5.7.1 Numerical results	100
	5.8	Chapter summary	102
CHAPTER 6		**APPLICATIONS OF GENOCCHI WAVELET-LIKE BASIS AND GENOCCHI POLYNOMIALS FOR SOLVING FPDES AND VOFPDES**	**103**
	6.1	Introduction	103
	6.2	Genocchi wavelet-like basis	104
		6.2.1 Function approximation	104
	6.3	Error estimation	108
	6.4	Application of Genocchi wavelet-like for two- dimensional time-fractional convection diffusion equaion with variable coefficient	110
		6.4.1 Numerical results	111
	6.5	Application of Genocchi wavelet-like basis for two-dimensional time-fractional diffusion-wave equation with damping	115
		6.5.1 Numerical results	116
	6.6	Application of Genocchi wavelet-like basis for variable order two-dimensional time-fractional mobile-immobile advection-dispersion model	119
.		6.6.1 Numerical results	120
	6.7	Application of Genocchi wavelet-like basis for variable-order two-dimensional space-time fractional wave equation	123

	6.7.1 Numerical results	124
	6.8 Ritz-Galerkin method by using Genocchi polynomials for two-dimensional FPDEs	128
	6.8.1 Detail of satisfier function	129
	6.9 Error bound	131
	6.10 Application of Genocchi polynomials with Ritz-Galerkin method for two-dimensional time-fractional diffusion-wave equation with damping	133
	6.11 Numerical results	133
	6.12 Chapter summary	136
CHAPTER 7	**CONCLUSIONS AND FUTURE WORKS**	**138**
	7.1 Introduction	138
	7.2 Research Summary	138
	7.3 Conclusions	140
	7.4 Future works	142
	REFERENCES	**143**
	APPENDIX A	**156**
	VITA	**158**

LIST OF TABLES

2.1	Summary of Legendre polynomials and Legendre wavelets method and its applications	15
2.2	Summary of Chebyshev polynomials and Chebyshev wavelets method and its applications	17
2.3	Summary of Bernoulli polynomials and Bernoulli wavelets method and its applications	19
2.4	Summary of spectral Collocation method and its applications	21
2.5	Summary of Tau method and its applications	22
3.1	For $0 \leq m, l \leq 5$, the first few Bernoulli and Euler numbers	33
3.2	For $0 \leq m, l \leq 5$, the first few Bernoulli and Euler polynomials	34
4.1	Absolute errors of Example 4.5.1 for $\alpha = 1.85$ and $M = 4,6,8$	70
4.2	Comparison of the absolute errors of Example 4.5.1with equally space CGL collocation points for $\alpha = 1.85$ and $M = 5$	70
4.3	Comparison of the absolute errors of Example 4.5.2 with NPL in Ghandehari and Ranjbar (2013) for $\alpha = 0.5$, $M = 4$ and $t = 0.1, 0.5$	71
4.4	Comparison of the absolute errors of Example 4.5.2 with NPL in Ghandehari and Ranjbar (2013) for $\alpha = 0.5, t = 1$ and $M = 4$	72

4.5	Comparison of the maximum absolute errors of Example 4.5.2 with NPL in Ghandehari and Ranjbar (2013) for $\alpha = 0.5$ and $M = 6$	72
4.6	Absolute errors of Example 4.6.1 for different values of α and $M = 3$	74
4.7	Comparison of the absolute errors of Example 4.6.1 with Chebyshev wavelets method in Zhou and Xu (2016) for different values of α and $M = 6$	75
4.8	Comparison of the absolute errors of Example 4.6.2 with Sinc–Legendre collocation method in Saadatmandi *et al.* (2012) for $\alpha = 0.5, t = 0.5$, $M = 4$, $M = 6$ and $M = 8$	76
4.9	Comparison of the absolute errors of Example 4.6.2 with Sinc–Legendre collocation method in Saadatmandi *et al.* (2012) for $\alpha = 0.5, t = 0.5$, $M = 4$, $M = 6$ and $M = 8$ (continued)	76
4.10	Comparison of the obtained solutions of Example 4.6.2 with Sumudu decomposition method in Al-Khaled (2015) for $\alpha = 0.75$	77
4.11	Comparison of the obtained solutions of Example 4.6.2 with Sumudu decomposition method in Al-Khaled (2015) for $\alpha = 0.95$	77
4.12	Comparison of maximum absolute errors of Example 4.7.1 with FEM in Dehghan *et al.* (2016a) for $\alpha = 1.95$ and different values of M	80
4.13	Comparison of maximum absolute errors L_∞ for $\sigma = 0.5, \gamma = 0.2$ of Example 4.8.1 with Legendre spectral element method in Dehghan *et al.* (2016b) for different values of M	83
4.14	L_2 for $\sigma = 0.3, \gamma = 0.7$ of Example 4.8.1 for Different values of M	83

5.1 Comparison of the obtained solutions of Example 5.5.1 with Sumudu decomposition method in Al-Khaled (2015) for $\alpha = 0.75$ 96

5.2 Comparison of the obtained solutions of Example 5.5.1 with Sumudu decomposition method in Al-Khaled (2015) for $\alpha = 0.95$ 96

5.3 Comparison of the obtained solutions of Example 5.6.1with ADM and VIM in Momani and Odibat (2006) for $\alpha = 1.5$ 98

5.4 Comparison of the obtained solutions of Example 5.6.1with ADM and VIM in Momani and Odibat (2006) for $\alpha = 1.75$ 99

5.5 Comparison of the obtained solutions of Example 5.7.1 with ADM and VIM in Momani and Odibat (2006) for $\alpha = 0.5$ 101

5.6 Comparison of the obtained solutions of Example 5.7.1 with ADM and VIM in Momani and Odibat (2006) for $\alpha = 0.75$ 101

6.1 Comparison of the absolute errors of Example 6.4 .1 with the results in Zhou and Xu (2016), Saadatmandi *et al.* (2012) and Chen *et al.* (2010) for $t = 0.5,\ \alpha = 0.5,\ k = 1,\ M = 3(\Delta = 2)$ 112

6.2 Comparison of the absolute errors of Example 6.4.2 with Chebyshev wavelets Collocation method in Zhou and Xu (2016) for different values of α, $k = 1$ and $M = 6\ (\Delta = 5)$ 113

6.3 Comparison of the absolute errors of Example 6.4 .2 with Chebyshev wavelets Collocation method in Zhou and Xu (2016) for $\alpha = 0.9, k = 1$ and $M = 6\ (\Delta = 5)$ 114

6.4 Comparison of the absolute errors of Example

	6.4.3 with Chebyshev wavelets Collocation method in Zhou and Xu (2016) for different values of $\alpha, k = 1$ and $M = 5$ ($\Delta = 4$)	115
6.5	Comparison of the absolute errors of Example 6.4.3 with Chebyshev wavelets Collocation method in Zhou and Xu (2016) for $\alpha = 0.9, k = 1$ and $M = 5$ ($\Delta = 4$)	115
6.6	Comparison of the absolute errors of Example 6.5.1 with Legendre wavelets method in Heydari *et al.* (2015) for different values of α, $k = 1$ and $M = 4$($\Delta = 3$)	117
6.7	Comparison of the absolute errors of Example 6.5.1 with Legendre wavelets method in Heydari *et al.* (2015) for $\alpha = 1.9$, $k = 1$ and $M = 4$($\Delta = 3$)	118
6.8	Comparison of the absolute errors of Example 6.5.2 with Legendre wavelets method in Heydari *et al.* (2015) for different values of α, $k = 1$ and $M = 5$($\Delta = 4$)	119
6.9	Comparison of the absolute errors of Example 6.5.2 with Legendre wavelets method in Heydari *et al.* (2015) for $\alpha = 1.9$, $k = 1$ and $M = 5$($\Delta = 4$)	119
6.10	Comparison of obtained solution of Example 6.6.1 with the exact solution for $k = 1$ and $M = 6$ ($\Delta = 5$)	121
6.11	Absolute errors of Example 6.7.1 with spectral Collocation method in Bhrawy *et al.* (2016)	125
6.12	Comparison of the absolute errors of Example 6.7.1 with NSFDM in Sweilam and Assiri (2015) for $k = 1, M = 5$	126

6.13 Maximum absolute errors of Example 6.7.1 in Bhraway *et al.* (2016) for different values of t 127

6.14 Comparison of the maximum absolute errors of Example 6.7.1 with the results in Sweilam and Assiri (2015) for different values of t and $M = 5$ 127

6.15 Comparison of the maximum absolute errors of Example 6.7.2 with NSFDM and SFDM in Sweilam and Assiri (2015) 128

6.16 Comparison of maximum absolute errors of Example 6.11.1 with the results in Reutskiy (2017), Bhrawy *et al.* (2015a), Liu *et al.* (2013) for $\alpha = 1.85$ 135

6.17 Comparison of the absolute errors of Example 6. 11.1with the results in Heydari *et al.* (2015) for different values of α and $M = 3$ 135

6.18 Comparison of the absolute errors of Example 6. 11.1 with the results in Heydari *et al.* (2015) for different values of $\alpha = 1.9$ and $M = 3$ 135

LIST OF FIGURES

4.1	First few Genocchi polynomials	53
4.2	CGL collocations points for $M = 4$	55
4.3	CGL collocations points for $M = 5$	56

LIST OF SYMBOLS AND ABBREVIATIONS

α	–	Alpha
β	–	Beta
γ	–	Gamma
μ	–	Mu
λ	–	Lambda
η	–	Eta
ϖ	–	Pi Variant
ω	–	Omega
ζ	–	Zeta
θ	–	Theta
ϕ	–	Phi Variant
ψ	–	Psi
Δ	–	Delta
ρ	–	Rho
σ	–	Sigma
Φ	–	Phi
ξ	–	Xi
τ	–	Tau
ϑ	–	Theta Variant
$\otimes$	–	Kronecker product
H_*^σ	–	Operational matrix of fractional derivative
$U_{M\times M}^\sigma$	–	Genocchi polynomials operational matrix of fractional derivative
K	–	Coefficient vector or expansion coefficient vector for

		two variables
$\overline{K}$	–	Coefficient or expansion coefficient vector for three variables
$\varpi(x,t)$	–	Satisfier function
$\sigma_{i,j}$	–	Expansion coefficient of fractional derivative
D^{α}	–	Fractional order derivative
D_t^{α}	–	Caputo's fractional derivative
D_{b-}^{α}	–	Right-sided Riemann-Liouville fractional derivative
D_{a+}^{α}	–	Left-sided Riemann-Liouville fractional derivative
${}_{RL}D_{t,t_R}^{\gamma(t)}$	–	Right Riemann-Liouville fractional derivative of variable order
${}_{RL}D_{t_L,t}^{\gamma(t)}$	–	Left Riemann-Liouville fractional derivative of variable order
${}_{RL}D_{t,t_R}^{-\gamma(t)}$	–	Right Riemann-Liouville fractional integral of variable order
${}_{RL}D_{t_L,t}^{-\gamma(t)}$	–	Left Riemann-Liouville fractional integral of variable order
${}_{C}D_{t,t_R}^{\gamma(t)}$	–	Right Riemann-Liouville fractional derivative of variable order
${}_{C}D_{t_L,t}^{\gamma(t)}$	–	Left Riemann-Liouville fractional derivative of variable order
$I_{\mathrm{a}+}^{\alpha}$	–	Left Riemann-Liouville fractional integral
$I_{\mathrm{b}-}^{\alpha}$	–	Right Riemann-Liouville fractional integral
$G_r(x)$	–	Genocchi polynomials of degree r
$B_m(x)$	–	Bernoulli polynomials of degree m
$E_l(x)$	–	Euler polynomials of degree l
$\Gamma(.)$	–	Euler Gamma function
Ω	–	Bounded domain
M	–	Scale level of approximation
$\psi(x)$	–	An oscillatory function

DE	–	Differential equation
FC	–	Fractional calculus
FDEs	–	Fractional differential equations
PDE	–	Partial differential equation
FPDEs	–	Fractional partial differential equations
TFPDES	–	Time fractional partial differential equation
SFPDE	–	Space fractional partial differential equation
STFPDE	–	Space-time fractional partial differential equation
VOFPDEs	–	Variable-order fractional partial differential equations
ADM	–	Adomian decomposition method
LADM	–	Laplace Adomian decomposition method
VIM	–	Variational iteration method
LVIM	–	Laplace Variational iteration method
RVIM	–	Reconstruction of Variational iteration method
HPM	–	Homotopy perturbation method
HPTM	–	Homotopy perturbation transform method
HAM	–	Homotopy analysis method
CGL	–	Chebyshev Gauss Lobatto
SL-GL-C	–	Shifted Legendre Gauss-Lobatto collocation method
FIDEs	–	Fractional integro-differential equations
SLOM	–	Shifted Legendre operational matrix
NSFDM	–	Non-standard finite difference method
SFDM	–	Standard finite difference method
MWR	–	Method of weighted residuals
SLC	–	Shifted Legendre Collocation method
FBE	–	Fractional Burgers' Equation
KDV	–	Korteweg de Vries equation
GFBWFs	–	Fractional-order Bernoulli wavelet functions
Q-SLT	–	Quadrature Shifted Legendre Tau method
KV	–	Kelvin-Voigt equation
GFLPs	–	Generalized fractional Legendre polynomials

NLP	–	Nonlinear programming problem

LIST OF APPENDICES

APPENDIX	TITLE	PAGE
A	Maple code for solving two-dimensional FPDEs	156

LIST OF PUBLICATIONS

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CHAPTER 1

INTRODUCTION

This chapter is comprised of some preliminaries given in section 1.1 that have been used in this research work. The reason to conduct this research is illustrated in section 1.3. To achieve the research aim, four objectives have been set in section 1.4. The scope of research and the main contribution are discussed in section 1.5 and section 1.6 respectively. Section 1.7 consists of thesis organization.

1.1 Preliminaries

In this section, some basic definitions of FPDEs, mathematical solution, solution methods and method of weighted residuals (MWR) are explained.

1.1.1 Fractional partial differential equations

FPDEs are the generalization of classical partial differential equations (PDEs) with the fractional order derivatives D^{α}. The general form of FPDEs (Al-Khaled, 2015) can be written as

$$D_t^{\alpha}u(x,t) = Lu(x,t) + Nu(x,t) + g(x,t),\ m-1 < \alpha \leqslant m, \tag{1.1}$$

where $u(x,t)$ is the unknown function, L is the linear operator, N is the general nonlinear operator and $g(x,t)$ is the source term. Similarly if the fractional order derivative is replaced with the variable order derivative then the equation would be

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